Ex. 2.2, 2.4, 2.6, 2.10

2.2 (a) Let $X$ denote the time (days) to tumor development.

The survival function of the Weibull distribution with $\alpha = 2$ and $\lambda = 0.001$ (using Table 2.2):

$$S(x) = \exp(-0.001x^2)$$

The probability that a rat will be tumor free at 30 days (in other words, the time the rat develops a tumor is greater than 30 days) is

$$S(30) = \exp(-0.001(30)^2) = .407$$

For $x = 45$ days,

$$S(45) = \exp(-0.001(45)^2) = .132$$

For $x = 60$ days,

$$S(60) = \exp(-0.001(60)^2) = .027$$

2.2 (b) From Table 2.2, the mean time to tumor with $\alpha = 2$ and $\lambda = 0.001$ is

$$E(X) = \frac{\Gamma(1 + 1/2)}{(0.001)^{1/2}}$$

Note that $\Gamma(1 + x) = x\Gamma(x)$. Then,

$$E(X) = \frac{\Gamma(1 + 1/2)}{(0.001)^{1/2}} = \frac{0.5\Gamma(0.5)}{(0.001)^{1/2}} = \frac{0.5\sqrt{\pi}}{(0.001)^{1/2}} = 28.02 \text{ days}$$

2.2 (c) Hazard rate function:

$$h(x) = \alpha \lambda x^{\alpha-1}$$

For $x = 30$ days,

$$h(30) = 2(0.001)(30)^{2-1} = .06$$

For $x = 45$ days,
\( \hat{h}(45) = 2(0.001)(45)^{2-1} = 0.09 \)

For \( x = 60 \) days,

\[ \hat{h}(60) = 2(0.001)(60)^{2-1} = 0.12 \]

**2.2 (d)** The probability of survival, \( 1 - p \), at time \( x_p \) is

\[ 1 - p = S(x_p) = \exp(-0.001x_p^2) \]

Solving for \( x_p \),

\[ \ln(1 - p) = -0.001x_p^2 \]

\[ x_p = \left[ -\frac{\ln(1 - p)}{0.001} \right]^{1/2} \]

Therefore, the median time to tumor is

\[ x_{50} = \left[ -\frac{\ln(0.5)}{0.001} \right]^{1/2} = 26.32 \text{ days} \]

**2.4 (a)** Show that the hazard rate has a bathtub shape and find the time at which the hazard rate changes from decreasing to increasing.

The hazard rate for the exponential power distribution is

\[ \hat{h}(x) = \frac{f(x)}{S(x)} = -\frac{d\ln(S(x))}{dx} = \frac{d\{1 - \exp[(\lambda x)^\alpha]\}}{dx} = \alpha \lambda\alpha x^{\alpha-1} \exp[(\lambda x)^\alpha] \]

For \( \alpha = 0.5 \),

\[ \hat{h}(x) = 0.5 \lambda^{0.5} x^{-0.5} \exp[(\lambda x)^{0.5}] \]

The hazard rate for various values of \( \lambda \) is plotted as follows. The hazard rate has a bathtub shape for positive \( \lambda \). Here are several examples:
\[ h'(x) = 0.5(0.5 - 1)\lambda^{0.5}x^{0.5-2} \exp[(\lambda x)^{0.5}] + (0.5\lambda^{0.5}x^{0.5-1})^2 \exp[(\lambda x)^{0.5}] \]
\[ = 0.25 \exp[(\lambda x)^{0.5}] \lambda^{0.5}x^{-1.5}(\lambda x)^{0.5} - 1). \]

For \( x < 1/\lambda, \lambda x - 1 < 0, \) so \( h'(x) < 0, \) i.e. \( h(x) \) is decreasing here. For \( x > 1/\lambda, \lambda x - 1 > 0, \) so \( h'(x) > 0, \) i.e. \( h(x) \) is increasing here.

To find the time at which the hazard rate changes from decreasing to increasing for \( \alpha < 1, \) find the local minimum:

\[ h'(x) = \alpha(\alpha - 1)\lambda^{\alpha}x^{\alpha-2} \exp[(\lambda x)\alpha] + (\alpha\lambda^{\alpha}x^{\alpha-1})^2 \exp[(\lambda x)\alpha] = 0 \]
\[ \alpha(\alpha - 1)\lambda^{\alpha}x^{\alpha-2} + (\alpha\lambda^{\alpha}x^{\alpha-1})^2 = 0 \]
\[ \alpha\lambda^{2}x^2 = 1 - \alpha \]
\[ x = \left(\frac{1 - \alpha}{\alpha\lambda^{2}}\right)^{1/2} \]

For \( \alpha = 0.5, \)
\[ x = \frac{1}{\lambda}. \]

2.4 (b) If \( \alpha = 2, \) show that the derivative of the hazard rate is positive for all \( x > 0. \)

\[ h'(x) = \exp[(\lambda x)^{2}] [\alpha(\alpha - 1)\lambda^{2}x^{\alpha-2} + (\alpha\lambda^{2}x^{\alpha-1})^2] \]

Since \( \exp[(\lambda x)^{2}] \geq 0 \) always, the other quantity in square brackets must be shown to be positive for all \( x \geq 0. \) For \( \alpha = 2, \)
\[ 2\lambda^{2} + (2\lambda^{2}x^{-1})^2 = 2\lambda^{2}(1 + 2\lambda^{2}x^{2}) \]
which is positive for all $x \geq 0$. Therefore, the hazard rate of $x$ is monotone increasing.

2.6 (a) Let $X$ denote the time to death (in months) of a randomly chosen mouse. The survival function of the Gompertz distribution:

$$S(x) = \exp \left[ \frac{\theta}{\alpha} \left(1 - e^{\alpha x}\right) \right]$$

For $\theta = 0.01$ and $\alpha = 0.25$,

$$S(x) = \exp[0.04(1 - e^{0.25x})]$$

The probability that a randomly chosen mouse will leave at least one year is the probability it is still alive at one year (12 months):

$$S(12) = \exp[0.04(1 - e^{0.25(12)})] = 0.466$$

2.6 (b) The probability that a randomly chosen mouse will die within the first six months is

$$1 - S(6) = 1 - \exp[0.04(1 - e^{0.25(6)})] = 0.130$$

2.6 (c) The probability of survival, $1 - p$, at time $x_p$ is

$$1 - p = S(x_p) = \exp[0.04(1 - e^{0.25x_p})]$$

Solving for $x_p$,

$$\ln(1 - p) = 0.04(1 - e^{0.25x_p})$$

$$x_p = 4\ln \left[ 1 - \frac{\ln(1 - p)}{0.04} \right]$$

Therefore, the median time to tumor is

$$x_{30} = 4\ln \left[ 1 - \frac{\ln(0.5)}{0.04} \right] = 11.63 \text{ months}$$
**Problem: 1.** From equation 2.3.4 we know \( S(y) = \exp \left[ - \int_0^y h(u) \, du \right] \), so \( S(y+t) = \exp \left[ - \int_0^{y+t} h(u) \, du \right] \).

For \( t \geq 0 \), \( P[Y > y+t \mid Y > y] = P[Y > y+t] / P[Y > y] = S(y+t)/S(y) \) so from above we now have:

\[
\frac{S(y+t)}{S(y)} = \exp \left[ - \int_0^{y+t} h(u) \, du \right] = \exp \left[ - \int_0^y h(u) \, du - \int_y^{y+t} h(u) \, du \right] = \exp \left[ - \int_y^{y+t} h(u) \, du \right]
\]

so \( P[Y > y+t \mid Y > y] = \exp \left[ - \int_y^{y+t} h(u) \, du \right] \)

Given \( h(y) \) is an increasing function. \( S(y+t)/S(y) = \exp \left[ - \int_0^{y+t} h(u) \, du \right] \)

\[
\ln[S(y+t)/S(y)] = - \int_0^{y+t} h(u) \, du - \int_0^y h(u) \, du \] and thus \( -\int \ln[S(y+t)/S(y)]/dy = h(y+t)-h(y) \)

When \( h(y+t)-h(y) > 0 \) then \( -\int \ln[S(y+t)/S(y)]/dy > 0 \), or \( \int \ln[S(y+t)/S(y)]/dy < 0 \)

Thus \( P[Y > y+t \mid Y > y] \) is a decreasing function when \( h(y) \) is an increasing function.

**Problem: 2.** Given the lifetime variable \( Y \) is exponentially distributed with a positive constant \( \beta \),

we know from Table 2.2 that \( S(y) = e^{-y/\beta} \) for \( y \geq 0 \). Therefore for \( t \geq 0 \) we can write:

\[
S(y+t) = e^{-(y+t)/\beta} = e^{-y/\beta} \cdot e^{-t/\beta} = S(y) \cdot S(t) \]

Thus a lifetime variable that is exponentially distributed exhibits the memoryless property: \( S(y+t) = S(y)S(t) \), \( y \geq 0, t \geq 0 \)