Midterm 2 Take-Home

Direct proof that \( \bar{X} \) and \( S^2 \) are independent when sampling from the \( N(\mu, \sigma^2) \) distribution.

Let \( X_1, X_2 \) independent \( N(\mu, \sigma^2) \) random variables. (A random sample of size \( n = 2 \).)

1. Show that \( Y_1 = X_1 + X_2 \) and \( Y_2 = X_2 - X_1 \) are independent.

2. What is the distribution of \( Y_1 \)? What is the distribution of \( Y_2 \)?

3. Show that \( W_1 = \frac{1}{2} Y_1 \) and \( W_2 = \frac{1}{2} Y_2 \) are independent.

   Note:
   \[
   W_1 = \frac{1}{2} Y_1 = \frac{1}{2} (X_1 + X_2) = \bar{X}
   \]

   and
   \[
   W_2 = \frac{1}{2} Y_2 = \frac{1}{2} (X_2 - X_1) = \frac{1}{2} X_2 - \frac{1}{2} X_1
   \]

   \[
   = X_2 - \frac{1}{2} X_1 - \frac{1}{2} X_2
   \]

   \[
   = X_2 - \left( \frac{X_1 + X_2}{2} \right) = X_2 - \bar{X}
   \]

   (So \( \bar{X} \) and the \( n - 1 \) deviations from the sample mean are independent.)

4. What is the distribution of \( W_1 \)? What is the distribution of \( W_2 \)?

5. Show that since \( W_1 \) and \( W_2 \) are independent that \( W_3 = X_1 - X \) is also independent of \( W_1 \).

   Note:
   \[
   X_1 - \bar{X} + X_2 - X = 0
   \]

   (So \( \bar{X} \) and the first deviation \( X_1 - \bar{X} \) are also independent.)

6. Argue that \( \bar{X} \) and \( S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 \) are independent for a random sample of size \( n = 2 \) from the \( N(\mu, \sigma^2) \) distribution.

7. What is the distribution of \( \bar{X} \)? What is the distribution of \( S^2 \)?

8. Develop the same results for \( n = 3 \).

9. Develop the same result for a sample of size \( n \).