## Unit 9: Contrasts of Sample Means

### 9.1. Definition of a Contrast

In designing an experiment with a balanced ANOVA model one often has in mind to estimate a linear combination $\Lambda = \mathbf{c}\mu^T = \Sigma_i c_i \mu_i$ of population means $\mu_i$, where the components of $\mathbf{c}$ add to 0, and sums are taken over all $a$ levels of the appropriate factor. (Here bold face indicates row vectors and superscript $^T$ indicates the transpose.)

The estimate of $\Lambda$ is $H = \mathbf{cm}^T$, where $\mathbf{m}$ is the row vector of $a$ sample means (y-bars based on $n$ observations each) that estimate the $a$ population means $\mu_i$. The quantity $H$, computable from experimental results, is called a contrast of sample means.

### 9.2. Confidence Intervals and Tests for Designed Contrasts

Consider a fixed-effects one-way ANOVA model with $n$ observations in each of $a$ groups. The variance of each observation is $\sigma_f^2$, estimated by $\text{MS(Error)} = \text{MSD}$, and the variance of $H$ is $\sigma_H^2 = \sigma_f^2 \mathbf{cc}^T / n$. We let $K = \mathbf{cc}^T = \Sigma_i c_i^2$ so that $\sigma_H^2 = K \sigma_f^2 / n$.

The variance of the contrast $H$ is estimated by $s_H^2 = K(\text{MSD}/n)$. Thus a 95% confidence interval for $\Lambda$ is $H \pm t^* s_H$, where $t^* = t(.025; \nu)$ and $\nu$ is the df of MSD.

The quantity $Q = nH^2 / K$ is called the component of SS(Factor) corresponding to the contrast $\Lambda$. An appropriate test of the hypothesis $H_0$: $\Lambda = 0$ against the two-sided alternative is to reject (5% level) when

$$Q/\text{MSD} = H^2 / s_H^2 > F^* = F(.05; 1, \nu).$$

Note that $t^*^2 = F^*$. Properly interpreted, the formula for $Q$ given above holds for a wide variety of ANOVA models. SS(Factor) may refer to any fixed factor in the model. Then MSD is the denominator mean square of the F-statistic for testing the significance of that factor, $\nu = \text{df(MSD)}$, and $n$ is the number of observations for each level of the factor (in more complex designs, including all cells that may be relevant).

**Example:** Refer to Example 15.8 in O/L 6e, p960. This is a randomized block design with $a = 3$ insecticides and $b = 4$ plots considered as blocks. The observations are numbers of seedlings. Insecticide means, based on $n = b = 4$ observations each, are 58 for Insecticide 1, 87 for Insecticide 2, and 80 for Insecticide 3. Table 15.9 p961 shows MS(Error) = 4.333 with $\nu = 6$ degrees of freedom.

Suppose that Insecticide 1 is of one chemical type and that Insecticides 2 and 3 are both of another type. Then one point of the experiment, *as originally designed*, may have been to compare types of insecticides. That is, we would look at the difference between Insecticide 1 and the average of Insecticides 2 and 3.
The appropriate contrast would be \( H = 58 - 0.5(87 + 80) = -25.5 \), so that the vector of components is \( \mathbf{c} = (1, -0.5, -0.5) \), giving \( K = 1 + 0.25 + 0.25 = 1.50 \). Hence,

\[
Q = bH^2 / K = (4)(-25.5)^2 / 1.50 = 1734.
\]

Because \( Q / \text{MS(Error)} = 1734 / 4.333 = 400.15 > F(0.05; 1, 6) = 5.987 \), we reject the null hypothesis that \( \Lambda = 0 \) and conclude that the two chemical types of insecticides differ.

In fact, because \( s_{H2}^2 = (1.50)(4.333) / 4 = 1.625 = (1.274)^2 \) and \( t^* = t(0.025; 6) = 2.447 \), we estimate that the difference between types is

\[
-25.5 \pm (2.447)(1.274) \text{ or } -25.5 \pm 3.13
\]

Notice that this 95% confidence interval does not include 0.

### 9.3. Orthogonal Contrasts

Infinitely many contrasts \( H \) may be formed using the treatment means of a factor if the coefficients \( c_i \) are restricted only in that they must sum to 0. Two contrasts \( H_1 \) and \( H_2 \) are said to be orthogonal if and only if \( \mathbf{c}_1 \mathbf{c}_2^T = \Sigma c_i c_{2i} = 0 \), where the subscripts of \( H_1 \) are \( c_{1i} \) and those of \( H_2 \) are \( c_{2i} \).

If a factor has \( a - 1 \) df, then one can find \( a - 1 \) orthogonal contrasts \( H_1, H_2, ..., H_{a-1} \) among the treatment means; these are called a complete set of orthogonal contrasts.

It can be shown that the sum \( \Sigma Q_j \) of the corresponding \( a - 1 \) components is equal to \( \text{SS(Factor)} \). We say that \( \text{SS(Factor)} \) has been partitioned into \( a - 1 \) orthogonal components.

**Example:** Returning to Example 15.8 in O/L 6e, we find that the contrast \( H_2 = 87 - 80 = 7.00 \) estimates the difference between the two insecticides of the second type (and ignores Insecticide 1). It has coefficients \( c_{21} = 0, c_{22} = 1, \text{ and } c_{23} = 1 \). The contrasts \( H_1 \) (called simply \( H \) in Section 9.2 above) and \( H_2 \) are orthogonal, because

\[
\mathbf{c}_1 \mathbf{c}_2^T = (1)(0) + (-0.5)(1) + (-0.5)(-1) = 0.
\]

There are \( a - 1 = 2 \) df for the Insecticide effect, so \( H_1 \) and \( H_2 \) are a complete set of orthogonal contrasts.

The component \( Q_2 \) of \( \text{SS(Insecticide)} \) corresponding to the contrast \( H_2 \) is

\[
Q_2 = (4)(7.00)^2 / 2 = 98.00.
\]

As anticipated,

\[
Q_1 + Q_2 = 1734.00 + 98.00 = 1832.00 = \text{SS(Insecticide)}.
\]

Do the two chemically similar insecticides (2 and 3) differ significantly in effect? Because

\[
Q_2 / \text{MS(Error)} = 98.00 / 4.333 = 22.6 > F(0.05; 1, 6) = 5.987,
\]

we conclude that they do.

Note: The above comparison of Insecticides 2 and 3 is equivalent to using Fisher’s HSD: Here \( \text{LSD} = t(0.025, 6)[4.333(2/4)]^{1/2} = 2.447(1.472) = 3.601 \). Thus the difference

\[
7.0 > 3.601 \text{ is significant. Equivalently, LSD test rejects because } 7.0/1.472 = 4.7554 > 2.447.
\]

Notice that \( 4.7554 = 22.6 \) and \( 2.447^2 = 5.987 \).
9.4. Testing Contrasts Suggested by the Data

In the case where a test of a contrast is suggested by looking at the data (rather than being one of very few contrasts established in advance as part of the experimental plan), more caution is warranted. Then \( H_0 \) is rejected at the 5% level if

\[
H^2 / s_{H}^2 = Q / MS_D > S^2 = (a - 1)F^*,
\]

where \( F^* = F(.05, a-1, \nu) \).

This is a relatively conservative procedure: The test statistic for Factor is \( MS(\text{Factor})/MS_D > F^* \) and \( Q / (a-1) \) cannot exceed \( MS(\text{Factor}) = SS(\text{Factor}) / (a-1) \). In effect, in order to test the contrast rather than the Factor, we replace \( SS(\text{Factor}) \) by one of its components to form the F-ratio. The Scheffé multiple comparison procedure is based on confidence intervals \( H \pm S_{sH} \).

**Example:** A standard physiology lab experiment at Cal State Hayward involves measuring the basal metabolism of mice at various temperatures. (Basal metabolism is volume of oxygen uptake adjusted for atmospheric pressure, length of time and the body volume of the subject.)

In one run of such an experiment, the metabolism of each of \( a = 4 \) randomly chosen mice was measured \( r = 5 \) times at each of \( b = 3 \) temperatures (28°C, 20°C, 12°C). This is a two-factor, mixed-model design (Temperatures fixed, Mice random). Data are shown below (also see Quiz Question 8 on the Statistics Department site.)

**METABOLIC RATES OF MICE AT THREE TEMPERATURES**

<table>
<thead>
<tr>
<th>Temperature</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>28°C</td>
<td>4.2</td>
<td>4.6</td>
<td>4.1</td>
<td>3.7</td>
</tr>
<tr>
<td></td>
<td>4.2</td>
<td>3.9</td>
<td>4.1</td>
<td>3.7</td>
</tr>
<tr>
<td></td>
<td>6.3</td>
<td>3.9</td>
<td>3.4</td>
<td>3.7</td>
</tr>
<tr>
<td></td>
<td>3.8</td>
<td>3.2</td>
<td>3.4</td>
<td>4.3</td>
</tr>
<tr>
<td></td>
<td>3.8</td>
<td>2.5</td>
<td>4.1</td>
<td>3.1</td>
</tr>
<tr>
<td>20°C</td>
<td>5.4</td>
<td>5.3</td>
<td>5.2</td>
<td>3.5</td>
</tr>
<tr>
<td></td>
<td>6.8</td>
<td>5.7</td>
<td>5.9</td>
<td>4.1</td>
</tr>
<tr>
<td></td>
<td>9.3</td>
<td>5.3</td>
<td>5.6</td>
<td>5.0</td>
</tr>
<tr>
<td></td>
<td>6.5</td>
<td>5.6</td>
<td>5.2</td>
<td>4.7</td>
</tr>
<tr>
<td></td>
<td>7.5</td>
<td>6.3</td>
<td>6.3</td>
<td>5.0</td>
</tr>
<tr>
<td>12°C</td>
<td>7.7</td>
<td>7.3</td>
<td>6.8</td>
<td>8.2</td>
</tr>
<tr>
<td></td>
<td>8.1</td>
<td>7.0</td>
<td>6.8</td>
<td>6.3</td>
</tr>
<tr>
<td></td>
<td>8.8</td>
<td>7.4</td>
<td>6.8</td>
<td>6.0</td>
</tr>
<tr>
<td></td>
<td>10.3</td>
<td>7.0</td>
<td>6.8</td>
<td>5.4</td>
</tr>
<tr>
<td></td>
<td>11.0</td>
<td>7.4</td>
<td>7.2</td>
<td>6.0</td>
</tr>
</tbody>
</table>

For these data, \( SS(\text{Temp}) = 123.589 \), \( MS(\text{Temp}) = 123.589/2 = 61.795 \), and \( MS(\text{Interaction}) = 1.283 \). The F-ratio for testing the Temperature effect, \( F = 61.795 / 1.283 = 48.16 \) has 2 and 6 df, and so is very highly significant.
The temperature means are 3.900 (28°C), 5.710 (20°C), and 7.415 (12°C), each based on \( n = ar = 20 \) observations. Suppose that one of the students, noticing the equal temperature intervals (8°C between levels) and noticing the steadily increasing average metabolic rates in the data, speculates that there may be a linear relationship between metabolic rate and temperature. This would suggest evaluating the "linear component" contrast

\[ H_1 = (-1)(3.900) + (0)(5.710) + (1)(7.415) = 3.515 \]

based on \( c_1 = (-1, 0, 1) \), which has \( K = 2 \). Thus, \( Q_1 = 20(3.515)^2/2 = 123.552 \).

That the component \( Q_1 = 123.552 \) is nearly as large as \( \text{SS(Temp)} = 123.589 \) already suggests that \( H_1 \) is a highly significant contrast. That it is significantly different from 0 at the 5% level may be confirmed by noting that

\[ Q_1 / \text{MS(Interaction)} = 123.552 / 1.283 = 96.30 > 2F(.05; 2, 6) = (2)(5.14) = 10.28. \]

The "quadratic" contrast with \( c_2 = (1, -2, 1) \) is orthogonal to the linear one just discussed. We leave it to you to show that \( Q_1 + Q_2 = \text{SS(Temp)} \), and that \( H_2 \) does not differ significantly from 0.

If the student had read the theory behind the experiment ahead of time, had suspected that there might be a linear relationship between basal metabolism and temperature (in the range of this experiment), and had proposed before seeing the data to test \( H_1 \), then it would have been legitimate to perform the test on that contrast according to the procedure of Section 9.2. (Do this test and compare with the result above.)

9.5. From One-Way to Two-Way ANOVA Via Contrasts

In certain circumstances it is possible to derive a more complex design from a simpler one by use of orthogonal contrasts. This can provide a better understanding of both the more complex design and of the idea of orthogonal contrasts. Here we show how to use contrasts to derive a two-way ANOVA with interaction from an appropriate one-way design.

9.5.1. The Data

Consider an experiment in which 40 rats are divided at random into four groups of 10 each. Each group is fed a different diet for a period of time and the weight gain of each rat (in grams) is recorded. The data are as shown below:

<table>
<thead>
<tr>
<th>Diet</th>
<th>Weight Gain (g)—Ten Rats Per Diet Group</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>73 102 118 104 81 107 100 87 117 111</td>
<td>100.0</td>
</tr>
<tr>
<td>2</td>
<td>90 76 90 64 86 51 72 90 95 78</td>
<td>79.2</td>
</tr>
<tr>
<td>3</td>
<td>98 74 56 111 95 88 82 77 86 92</td>
<td>85.9</td>
</tr>
<tr>
<td>4</td>
<td>107 95 97 80 98 74 74 67 89 58</td>
<td>83.9</td>
</tr>
</tbody>
</table>

Data taken from a larger study reported in Snedecor and Cochran: *Statistical Methods, 7th ed.*, Chapter 16 (Factorial Designs), Iowa State University Press, 1980, Ames, IA.
9.5.2. One-Way Analysis of Variance

A one-way ANOVA shows that there are significant differences among the diet groups as to weight gain.

```
Analysis of Variance for WtGain

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diet</td>
<td>3</td>
<td>2404.1</td>
<td>801.4</td>
<td>3.58</td>
<td>0.023</td>
</tr>
<tr>
<td>Error</td>
<td>36</td>
<td>8049.4</td>
<td>223.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>39</td>
<td>10453.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
```

9.5.3. Contrasts

Standard multiple comparison methods could be used to explore the patterns of differences present. (Use Fisher's LSD to find this pattern of differences.) Here we investigate several contrasts among the group means based on additional information about the diets.

Actually, the protein in these diets are of two different Kinds: Diets 1 and 2 were based on beef, while Diets 3 and 4 on cereal. Furthermore, the diets contained different Amounts of protein: Diets 1 and 3 had a high level protein, while Diets 2 and 4 were low in protein.

Because the Diet effect has three degrees of freedom, a complete set of three contrasts can be specified. Suppose that, knowing the characteristics of the four diets, the experimenter designated three "design" contrasts before seeing the data. The vectors of coefficients and rationale are as follows:

- $(1, 1, -1, -1)$: Compares Beef against Cereal ("Kind")
- $(1, -1, 1, -1)$: Compares High against Low ("Amount")
- $(1, -1, -1, 1)$: Interaction between Kind and Amount

Also, after seeing that Beef at the High protein level gave the best results among the four treatments, suppose that the experimenter decided to consider an additional ad-hoc contrast:

- $(3, -1, -1, -1)$: Compares Beef/High against all other diets.

In terms of the notation developed earlier $n = 10$ and $MSD = MS(Error) = 223.6$. Computations of components $Q$ and tests of the significance of these contrasts are summarized in the following table:

```
<table>
<thead>
<tr>
<th>Status</th>
<th>Coefficients</th>
<th>$K$</th>
<th>$H$</th>
<th>$Q$</th>
<th>$Q/MSD$</th>
<th>Critical Value</th>
<th>Signif @ 5%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-chosen</td>
<td>$(1, 1, -1, -1)$</td>
<td>4</td>
<td>9.4</td>
<td>220.9</td>
<td>0.99</td>
<td>F(.05;1,36)= 4.11</td>
<td>No</td>
</tr>
<tr>
<td>(orthogonal set)</td>
<td>$(1, -1, 1, -1)$</td>
<td>4</td>
<td>22.8</td>
<td>1299.6</td>
<td>5.81</td>
<td>4.11</td>
<td>Yes</td>
</tr>
<tr>
<td></td>
<td>$(1, -1, -1, 1)$</td>
<td>4</td>
<td>18.8</td>
<td>883.6</td>
<td>3.95</td>
<td>4.11</td>
<td>&quot;Nearly&quot;</td>
</tr>
<tr>
<td>Ad hoc</td>
<td>$(3, -1, -1, -1)$</td>
<td>12</td>
<td>51.0</td>
<td>2167.5</td>
<td>9.69</td>
<td>3F(.05;3,36)=8.60</td>
<td>Yes</td>
</tr>
</tbody>
</table>
```
The designed contrasts show a significant Amount effect and suggest the possibility of interaction. The ad hoc contrast is also significant, even though it is judged against a more stringent standard. [If the first contrast is expressed as (.5, .5, –.5, –.5), then find its component $Q_1$. Is it still 220.9?]

9.5.4. Two-Way Analysis of Variance

In order to analyze this experiment directly as a two-way ANOVA using Minitab, we create columns of subscripts for Kind (with Beef=1 and Cereal=2) and for Amount (with High=1 and Low=2).

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kind</td>
<td>1</td>
<td>220.9</td>
<td>220.9</td>
<td>0.99</td>
<td>0.327</td>
</tr>
<tr>
<td>Amount</td>
<td>1</td>
<td>1299.6</td>
<td>1299.6</td>
<td>5.81</td>
<td>0.021</td>
</tr>
<tr>
<td>Kind*Amount</td>
<td>1</td>
<td>883.6</td>
<td>883.6</td>
<td>3.95</td>
<td>0.054</td>
</tr>
<tr>
<td>Error</td>
<td>36</td>
<td>8049.4</td>
<td>223.6</td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>39</td>
<td>10453.5</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Notice that the Diet effect (with 3 df) of the one-way ANOVA shown previously has been resolved into three effects in this two-way ANOVA: Kind, Amount, and Interaction (each with 1 df). Verify that the SSs for these three effects sum to SS(Diet) in the one-way ANOVA.

The three designed contrasts also represent one degree of freedom each. They were chosen so each of them corresponds to one of the three effects in the two-way ANOVA. Verify that the components $Q$ of these three contrasts correspond to the SSs for the three effects, and that the tests of the contrasts correspond exactly to the F-tests of the three effects in the two-way ANOVA table.

Notice that the component 2167.5 for the ad hoc contrast is smaller than SS(Diet) = 2404.1 in the one-way ANOVA, and that the test of significance of that contrast uses the same standard as was used to test the Diet effect: divide by 3 and reject at the 5% level if the result exceeds $F(.05; 3, 36)$. 
9.5.5. Problems

The full dataset from which the data above were taken has two additional diets, each with 10 replications. Diet 5 consisted of Pork with a High protein level, Diet 6 of Pork with a Low protein level. (There were 60 rats in the full experiment.) The additional data are given in the last two rows of the table below:

<table>
<thead>
<tr>
<th>Diet</th>
<th>Weight Gain (g)—Ten Rats Per Diet Group</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1: Beef/High</td>
<td>73 102 118 104 81 107 100 87 117 111</td>
<td>100.0</td>
</tr>
<tr>
<td>2: Beef/Low</td>
<td>90 76 90 64 86 51 72 90 95 78</td>
<td>79.2</td>
</tr>
<tr>
<td>3: Cereal/High</td>
<td>98 74 56 111 95 88 82 77 86 92</td>
<td>85.9</td>
</tr>
<tr>
<td>4: Cereal/Low</td>
<td>107 95 97 80 98 74 74 67 89 58</td>
<td>83.9</td>
</tr>
<tr>
<td>5: Pork/High</td>
<td>94 79 96 98 102 102 108 91 120 105</td>
<td>99.5</td>
</tr>
<tr>
<td>6: Pork/Low</td>
<td>49 82 73 86 61 97 106 70 61 82</td>
<td>78.7</td>
</tr>
</tbody>
</table>


For your convenience, here is a text listing of the data in the table above:

73 102 118 104 81 107 100 87 117 111
90 76 90 64 86 51 72 90 95 78
98 74 56 111 95 88 82 77 86 92
107 95 97 80 98 74 74 67 89 58
94 79 96 98 102 102 108 91 120 105
49 82 73 86 61 97 106 70 61 82

1. Perform a one-way ANOVA on the six Diets. Use Fisher's LSD method to find the pattern of differences.
2. Find a complete set of 5 orthogonal contrasts corresponding to Kind (2 df: meats vs. cereal, beef vs. pork), Amount (1 df: high vs low), and Interaction (2 df). Partition the one-way SS(Diet) from Problem 1 into 5 corresponding parts. Test each of these contrasts as pre-chosen.
3. Perform a two-way ANOVA with Kind (3 levels), Amount (2 levels) and interaction. Show the connection between the components of your five contrasts and the SSs in the two-way ANOVA table.
4. Test the ad hoc contrast that compares the average of beef and pork at high levels with the average of the other four diets. Is this contrast significant at the 5% level?.

Acknowledgments and References

Measurements for the physiology lab experiment were made in a course taught in Winter 1997 at CSU East Bay (then CSU Hayward) by Professor R. Tullis and supplied by Mark Munneke, a student in the class. More complete discussions of contrasts may be found in the following three textbooks:


Note: The mathematical notation differs considerably among these authors. The notation used here is chosen to minimize confusion with notation in the references listed above, and to minimize the use of special mathematical symbols. Brownlee uses totals instead of averages in his formulas.