Statistics 6402 **Name:**___________________________
Winter 2006 – Dr. Kwon **FINAL EXAM**

Due Wed 3/15/06 6 PM. Submit to stat office mailbox.
For full credit, be sure to show clear and complete work.

1. (+20) Suppose that the weather in a City can be modeled as a Markov chain with two states: {sunny, rainy}, which are coded as {0, 1}, respectively. The transition matrix for the Markov chain is given below.

$$P = \begin{matrix} 0 \\ 1 \end{matrix}\begin{bmatrix} .9 & .1 \\ .6 & .4 \end{bmatrix}$$

a) If the weather is sunny today, what is the probability that it will be rainy tomorrow and the next day and then sunny on the following day? (+3)
b) If the weather today is sunny with probability **.**4 and rainy with probability **.**6, what is the probability that it will sunny today and tomorrow and rainy the next day? (+3)
c) If the weather today is sunny, what is the probability that it will be rainy two days from today? (+3)
d) Find the limiting distribution $\underline{\pi}$ by explicitly solving the appropriate system of equations. (+5)
e) What is the long-run expected fraction of time that the weather will be sunny? (+3)
f) If the current weather is sunny, what is the expected number of days until the weather returns to sunny? (+3)

2. (+15) Spam e-mails arrive at a professor's mailbox according to a Poisson process $\{N(t): t \geq 0\}$ with rate λ=3 per hour.
a) Find the probability that there are no spam emails in a two-hour period. (+3)
b) Find the probability (not two probabilities!) that there are two spam emails between 1 p.m. and 2 p.m. and five spam emails between 1 p.m. and 4 p.m., on the same day. (+3)
c) Let T be the time in hours between two successive spam emails. Find the P(T > 1 hour). (+3)
d) Let W_2 be the waiting time in hours from some time 0 to the second spam email. Find P(W_2 > 1 hour). (+3)
e) Find the mean and variance of W_2. (+3)

3. (+15) A barbershop has one barber and only one chair for waiting. Let λ be the arrival rate of customers and let μ be the service rate of the barber. If the shop is not full, then an arriving customer joins the system; if the shop is full, he or she leaves. Assume the system can be modeled as an M/M/1 queuing process with limited capacity (N = 2).
a) Write down the balance equations for the process. (See p.488) (+5)
b) Solve the balance equations to find P_0, P_1 and P_2. Write them in terms of $r = \lambda/\mu$. (+5)
c) If $r = \lambda/\mu = 1/5$, what fraction of time is the barber idle? (See P. 509) (+5)

4. (+30) On an entrance to a bridge, there is a waiting line of cars and a single toll booth. Assume this process can be modeled as an M/M/1 queue. Vehicles arrive at rate λ=6 per minute (or 300 vehicles per hour) and the toll booth has a service rate of μ = 6 per minute.
a) Find the probability that there is at most one vehicle in the system. (+3)
b) Find the average number of vehicles in the system. (+3)
c) Find the average waiting time of vehicles in the system. (+3)
d) Find the average waiting time of vehicles waiting in the queue. (+3)
e) Find the average number of vehicles waiting in the queue. (+3)
f) Suppose that another toll booth is added to the system so that we now have an M/M/2 queue. Furthermore, suppose that the service rate of each booth is μ = 6 per hour. Find out the steady state probability that # of vehicles in the system is 0, 1, 2, and 3, i.e. Evaluate P_0, P_1, P_2, and P_3. (See p. 530) (+5)

g) Consider the same situation as f). It turns out that the mean number of vehicles in the M/M/2 system in steady-state is 1.333. Use Little’s formula to find the mean time (in minutes) that a vehicle spends in this M/M/2 system. Then use Little’s formula again to find the new service rate μ’ of the single server in the original M/M/1 system so that the mean time that a customer spends in the M/M/1 system is the same as the mean time that a customer spends in the M/M/2 system. (+10)

* More on P. 530: the formula for M/M/k queue is written rather ambiguously. The correct formula is as follows:

$$P_i = \frac{\dfrac{(\lambda/\mu)^i}{i!}}{\displaystyle\sum_{j=0}^{k-1}\frac{(\lambda/\mu)^j}{j!} + \frac{(\lambda/\mu)^k}{k!}\frac{k\mu}{k\mu-\lambda}}, i \le k$$

. The formula for $i > k$ is OK.