1. (+20) Cars arrive at the drive-through of a bank at a rate of 9 cars per hour. There is only one service window and the mean service time is 2.5 minutes. Assume that this process can be modeled as an M/M/1 queue.

   a) Find the arrival and service rates $\lambda_A$ and $\lambda_S$, in number of cars per minute. (+4)

   b) Find $r = \frac{\lambda_A}{\lambda_S}$, which can be interpreted as the “traffic intensity”. (+2)

   c) Find the probability that the queue in steady-state will be idle when a car arrives at the drive-through. (+2)

   d) Find the expected value and standard deviation of the number of cars in the system in steady-state. (+4)

   e) Find the probability that two or more cars are in the system in steady-state. (+4)

   f) On average, how many minutes does a car spend in the drive-through? (+4)

2. (+20) A barbershop has one barber and only one chair for waiting. The expected time to serve a customer is 30 minutes and the customers arrive at the rate of one per hour. If the shop is not full, then an arriving customer joins the system; if the shop is full, he or she leaves. Assume this system can be modeled as a single-server Bernoulli queueing process with limited capacity ($C = 2$). Suppose that the frame size is 3 minutes.

   a) Find the arrival and service probabilities $p_A$ and $p_S$. (+4)

   b) Find the one-step transition matrix $P$ for this process. Check that each row sums to one. (+8)

   c) Find the steady-state probabilities $\pi_0$, $\pi_1$, and $\pi_2$. (+8)
3. (+20) Consider a setup that is similar to #2 above: A barbershop has one barber and only one chair for waiting. Let $\lambda_A$ be the arrival rate of customers and let $\lambda_S$ be the service rate of the barber. If the shop is not full, then an arriving customer joins the system; if the shop is full, he or she leaves. Assume this system can be modeled as a single-server Bernoulli queueing process with limited capacity ($C = 2$). If the frame size $\Delta$ is very small, then this process approximates an $M/M/1$ queueing process with limited capacity. Furthermore, the transition matrix for this Bernoulli process is approximately

$$
\begin{pmatrix}
1 - \lambda_d \Delta & \lambda_d \Delta & 0 \\
\lambda_s \Delta & 1 - (\lambda_d + \lambda_s) \Delta & \lambda_d \Delta \\
0 & \lambda_s \Delta & 1 - \lambda_s \Delta
\end{pmatrix}
$$

The limiting (steady-state) probability distribution exists by doing a simple calculation.

Using the above matrix, write out the three equations represented by $\pi = \pi P$. Observe that they are balance equations. At this point, don’t manipulate or solve the equations.

Now let $r = \lambda_A / \lambda_S$ and solve for $\pi_0$, $\pi_1$, and $\pi_2$ in terms of $r$.

If $r = 1/5$, what fraction of time is the barber idle?

4. (+20) At a bank, there is a waiting line and three tellers. Suppose that customers arrive at a rate $\lambda_A = 30$ per hour and each teller has a service rate of $\lambda_S = 15$ customers per hour. Assume that this process can be modeled as an $M/M/3$ queueing system. When doing the following parts, use fractional arithmetic throughout.

a) Check that the limiting (steady-state) probability distribution exists by doing a simple calculation.

b) Find the steady-state probabilities $\pi_0$ and $\pi_3$ as fractions.

c) Find the mean number of customers in the system in steady state using the following formula given in problem 7.7-8 of the textbook.

$$E(N) = r + \frac{r \pi_k / k}{(1 - r / k)^2}$$
Midterm #2 Solutions

1. 
   a) $\lambda_A = 9/60 = 0.15$ (cars per minute) and $\lambda_S = 1/2.5 = 0.4$ (cars per minute)
   b) $r = \lambda_A / \lambda_S = 0.15 / 0.4 = 0.375$
   c) $\pi_0 = (1-r) = 0.625$
   d) $E(N) = r / (1-r) = 0.375 / (1-0.375) = 0.6$
      $SD(N) = \sqrt{r / (1-r)} = \sqrt{0.375 / (1-0.375)} = 0.9797 = 0.980$
   e) $P(X > 1) = r^2 = 0.375^2 = 0.141$
   f) $E(T) = (r + 1) / (\lambda_A - \lambda_S) = 0.375 / (0.15 - 0.4) = 4$ (minutes)

2. 
   a) $p_A = 1/60 * 3 = 0.05$; $p_S = 1/30 * 3 = 0.1$
   b) $P = \begin{bmatrix} 0 & 1 - p_A & p_A & 0 \\ 1 - p_A & p_A & 0 & 0 \\ 0 & 1 - p_S & p_S & 0 \end{bmatrix} = \begin{bmatrix} 0.95 & 0.05 & 0 \\ 0.095 & 0.86 & 0.045 \\ 0 & 0.095 & 0.905 \end{bmatrix}$
      where * is found as the number that makes the row sum to be one.
   c) Solve $\pi P = \pi$ and $\pi 1 = 1$.
      $\pi_0 + 0.95 \pi_1 = \pi_0$;
      $(\pi_1 = 0.05/0.095 \pi_0 = 0.526 \pi_0)$
      $0.95 \pi_0 + 0.905 \pi_2 = \pi_1$;
      $(\pi_2 = 0.14/0.095 \pi_1 - 0.05/0.095 \pi_0 = 0.14/0.095 \pi_1 - 0.05/0.095 \pi_0 = 0.526 \pi_0)$
      $0.045 \pi_1 + 0.905 \pi_2 = \pi_2$;
      (redundant)
      $\pi_0 + \pi_1 + \pi_2 = 1$;
      $(\pi_0 + 0.526 \pi_0 + 0.250 \pi_0 = 1; \pi_0 = 1/(1 + 0.526 + 0.250) = 0.563)$
      Thus, $\pi_0 = 0.563$, $\pi_1 = 0.563 * 0.526 = 0.296$, $\pi_2 = 0.250 * 0.563 = 0.141$

3. 
   a) $(1 - \lambda_A \Delta) \pi_0 + \lambda_S \Delta \pi_1 = \pi_0$; $(\lambda_S \pi_1 = \lambda_A \pi_0)$ (**)
   $\lambda_A \Delta \pi_0 + (1 - (\lambda_A + \lambda_S)\Delta) \pi_1 + \lambda_S \Delta \pi_2 = \pi_1$;
   $(\lambda_A \pi_0 - \lambda_A \pi_1 - \lambda_S \pi_1 + \lambda_S \pi_2 = 0$, which reduces to $\lambda_S \pi_2 = \lambda_A \pi_1$ from (**))
   $\lambda_A \Delta \pi_1 + (1 - \lambda_S)\Delta \pi_2 = \pi_2$ (\lambda_S \pi_2 = \lambda_A \pi_1)
b) $\pi_1 = r \pi_0$
$\pi_2 = r \pi_1 = r^2 \pi_0$

Solving $\pi_0 + \pi_1 + \pi_2 = \pi_0 (1 + r + r^2) = 1$, we have:
$\pi_0 = \frac{1}{1 + r + r^2}$
$\pi_1 = r \pi_0$
$\pi_2 = r \pi_1 = r^2 \pi_0$

c) $\pi_0 = \frac{1}{1 + 1/5 + 1/5^2} = .806$

4. $\lambda_A = 30$, $\lambda_S = 15$, and M/M/3

a) $r = \lambda_A / \lambda_S = 2$; $r/k = 2/3 = .667 < 1$

b) $\pi_0 = \left[1 + \sum_{j=1}^{k-1} \frac{r^j}{j!} + \frac{r^k}{k!(1-r/k)} \right]^{-1} = (1 + 2/1 + 2^2/2! + 2^3/(3!(1 - 2/3)))^{-1} = (1 + 2 + 2 + 8/(3*2*(1/3)))^{-1} = 1/9$

$\pi_j = \begin{cases} 
\pi_0 \frac{r^j}{j!}, & j < k \\
\pi_0 \frac{r^j}{k!k^{j-k}} = \pi_k \left( \frac{r}{k} \right)^{j-k}, & j \geq k 
\end{cases}$, thus $\pi_3 = \pi_0 \frac{r^j}{k!k^{j-k}} = 1/9 \times 2^3/3! = 4/27$

c) $E(N) = r + \frac{r \pi_k}{(1 - r/k)^2} = 2 + 2* (4/27)/3 / (1-2/3)^2 = 2 + 8/9 = \frac{26}{9}$