Stat 6310 Spring 2006 Jaimie Kwon

Midterm #2 (100 minutes) **Name**____________________

- Be sure to show your work for full credit. Write your numerical answers in fractions or round them to the 3rd significant digits.
- Closed book, closed note. One two-sided, letter-sized, hand-written formula sheet allowed. A simple calculator is allowed but don't use it to do matrix multiplication or solving a system of equations.

1.(+20) Cars arrive at the drive-through of a bank at a rate of 9 cars per hour. There is only one service window and the mean service time is 2.5 minutes. Assume that this process can be modeled as an M/M/1 queue.

a) Find the arrival and service rates λ_A and λ_S, in number of cars per minute. (+4)

b) Find $r = \lambda_A/\lambda_S$, which can be interpreted as the "traffic intensity". (+2)

c) Find the probability that the queue in steady-state will be idle when a car arrives at the drive-through. (+2)

d) Find the expected value and standard deviation of the number of cars in the system in steady-state. (+4)

e) Find the probability that two or more cars are in the system in steady-state. (+4)

f) On average, how many minutes does a car spend in the drive-through? (+4)

2. (+20) A barbershop has one barber and only one chair for waiting. The expected time to serve a customer is 30 minutes and the customers arrive at the rate of one per hour. If the shop is not full, then an arriving customer joins the system; if the shop is full, he or she leaves. Assume this system can be modeled as a single-server Bernoulli queueing process with limited capacity (C = 2). Suppose that the frame size is 3 minutes.

a) Find the arrival and service probabilities p_A and p_S. (+4)

b) Find the one-step transition matrix P for this process. Check that each row sums to one. (+8)

c) Find the steady-state probabilities π_0, π_1, and π_2. (+8)

3.(+20) Consider a setup that is similar to #2 above: A barbershop has one barber and only one chair for waiting. Let λ_A be the arrival rate of customers and let λ_S be the service rate of the barber. If the shop is not full, then an arriving customer joins the system; if the shop is full, he or she leaves. Assume this system can be modeled as a single-server Bernoulli queueing process with limited capacity (C = 2). If the frame size Δ is very small, then this process approximates an M/M/1 queueing process with limited capacity. Furthermore, the transition matrix for this Bernoulli process is approximately

$$P \approx \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 1-\lambda_A\Delta & \lambda_A\Delta & 0 \\ \lambda_S\Delta & 1-(\lambda_A+\lambda_S)\Delta & \lambda_A\Delta \\ 0 & \lambda_S\Delta & 1-\lambda_S\Delta \end{bmatrix}$$

d) Using the above matrix, write out the three equations represented by $\underline{\pi} = \underline{\pi}P$. Observe that they are balance equations. At this point, don't manipulate or solve the equations. (+9)

e) Now let $r = \lambda_A / \lambda_S$ and solve for π_0, π_1, and π_2 in terms of r. (+9)

f) If $r = 1/5$, what fraction of time is the barber idle? (+2)

4.(+20) At a bank, there is a waiting line and three tellers. Suppose that customers arrive at a rate λ_A = 30 per hour and each teller has a service rate of λ_S = 15 customers per hour. Assume that this process can be modeled as an M/M/3 queueing system. When doing the following parts, use fractional arithmetic throughout.

a) Check that the limiting (steady-state) probability distribution exists by doing a simple calculation. (+5)

b) Find the steady-state probabilities π_0 and π_3 as fractions. (+10)

c) Find the mean number of customers in the system in steady state using the following formula given in problem 7.7-8 of the textbook. (+5)

$$E(N) = r + \frac{r\pi_k / k}{(1-r/k)^2}$$

Midterm #2 Solutions

1.

a) λ_A = 9/60 = **0.15 (cars per minute)** and λ_S = 1/2.5 = **0.4 (cars per minute)**

b) $r = \lambda_A/\lambda_S$ = 0.15/0.4 = **.375**

c) $\pi_0 = (1 - r)$ = **.625**

d) E(N) = r/(1-r) = .375/(1-.375) = **.6**;
SD (N) = sqrt(r)/(1-r) = sqrt(.375)/(1-.375) = .9797.. = **.980**

e) P(X > 1) = r^2 = .375^2 = **.141**

f) E(T) = $\left(\frac{r}{1-r}+1\right)\frac{1}{\lambda_S} = \frac{1}{(1-r)\lambda_S}$ =1/((1-.375)*.4) = **4 (minutes)**

2.

a) p_A = 1/60*3 = **.05**; p_S = 1/30*3 = **.1**

b)

$$P = \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 1-p_A & p_A & 0 \\ (1-p_A)p_S & * & (1-p_S)p_A \\ 0 & (1-p_A)p_S & * \end{bmatrix} = \begin{bmatrix} .95 & .05 & 0 \\ .095 & .86 & .045 \\ 0 & .095 & .905 \end{bmatrix}$$

where * is found as the number that makes the row sum to be one.

c) Solve $\boldsymbol{\pi}$ P = $\boldsymbol{\pi}$ and $\boldsymbol{\pi}$ **1** = 1.
$.95\ \pi_0 + .095\ \pi_1 = \pi_0$;
$(\pi_1 = .05/.095\ \pi_0 = .526\ \pi_0)$
$.05\ \pi_0 + .86\ \pi_1 + .095\ \pi_2 = \pi_1$;
$(\pi_2 = .14/.095\ \pi_1 - .05/.095\ \pi_0 = .14/.095 * .05/.095\ \pi_0 - .05/.095\ \pi_0 =$
$[.776 - .526]\ \pi_0 = .250\ \pi_0)$
$.045\ \pi_1 + .905\ \pi_2 = \pi_2$;
(redundant)
$\pi_0 + \pi_1 + \pi_2 = 1$;
$(\pi_0 + .526\ \pi_0 + .250\ \pi_0 = 1; \pi_0 = 1/(1 + .526 + .250) = 0.563)$;
Thus, π_0 = 0.563 , π_1 = 0.563*.526 = .296, π_2 = .250* .563= .141

3.

$$P \approx \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} \begin{bmatrix} 1-\lambda_A\Delta & \lambda_A\Delta & 0 \\ \lambda_S\Delta & 1-(\lambda_A+\lambda_S)\Delta & \lambda_A\Delta \\ 0 & \lambda_S\Delta & 1-\lambda_S\Delta \end{bmatrix}$$

a) $(1 - \lambda_A\Delta)\ \pi_0 + \lambda_S\Delta\ \pi_1 = \pi_0$; **($\lambda_S\ \pi_1 = \lambda_A\ \pi_0$) (**)**
$\lambda_A\Delta\ \pi_0 + (1 - (\lambda_A + \lambda_S)\Delta)\ \pi_1 + \lambda_S\Delta\ \pi_2 = \pi_1$;
($\lambda_A\ \pi_0 - \lambda_A\ \pi_1 - \lambda_S\ \pi_1 + \lambda_S\ \pi_2 = 0$, which reduces to **$\lambda_S\ \pi_2 = \lambda_A\ \pi_1$** from (**))
$\lambda_A\Delta\ \pi_1 + (1 - \lambda_S)\Delta\ \pi_2 = \pi_2$ **($\lambda_S\ \pi_2 = \lambda_A\ \pi_1$)**

b) $\pi_1 = r\,\pi_0$;
$\pi_2 = r\,\pi_1 = r^2\,\pi_0$
Solving $\pi_0 + \pi_1 + \pi_2 = \pi_0\,(1 + r + r^2) = 1$, we have:
$\boldsymbol{\pi_0 = 1/(1 + r + r^2)}$
$\boldsymbol{\pi_1 = r\,\pi_0}$**;**
$\boldsymbol{\pi_2 = r\,\pi_1 = r^2\,\pi_0}$

c) $\pi_0 = 1/(1 + 1/5 + 1/5\text{^}2) =$ **.806**

4. $\lambda_A = 30$, $\lambda_S = 15$, and M/M/3 ;

a) $r = \lambda_A / \lambda_S = 2$; **r/k = 2/3 = .667 < 1**

b) $\pi_0 = \left[1 + \sum_{j=1}^{k-1} \frac{r^j}{j!} + \frac{r^k}{k!(1 - r/k)}\right]^{-1}$ = (1 + 2/1 + 2^2/2! + 2^3/(3!(1 – 2/3)))$^{-1}$
= (1 + 2 + 2 + 8/(3*2*(1/3)))$^{-1}$ = **1/9**

$$\pi_j = \begin{cases} \pi_0 \dfrac{r^j}{j!}, & j < k \\ \pi_0 \dfrac{r^j}{k!k^{j-k}} = \pi_k \left(\dfrac{r}{k}\right)^{j-k}, & j \ge k \end{cases}$$
, thus $\pi_3 = \pi_0 \dfrac{r^j}{k!k^{j-k}}$ = 1/9*2^3/(3!) = **4/27**

c) $E(N) = r + \dfrac{r\pi_k / k}{(1 - r/k)^2}$ = 2 + 2* (4/27)/3 / (1-2/3)^2 = 2 + 8/9 = **26/9**