

Midterm #1 (100 minutes)

Name _____

- Be sure to show your work for full credit. Write your numerical answers in fractions or round them to the 3rd significant digits.
 - Closed book, closed note. One two-sided, letter-sized, hand-written formula sheet allowed. A simple calculator is allowed but don't use it to do matrix multiplication or solving a system of equations.
1. (+15) In each part, give the state space S and the one-step transition matrix of the described Markov chain $\{X(n) : n \geq 0\}$. Show some work.
- a) Three balls, numbered 1, 2, and 3, are distributed between two adjacent compartments, named A and B. A ball is selected at random from the three balls and is moved to the compartment that it is not currently in. This process is repeated indefinitely. Let $X(n)$ be the number of balls in compartment A after the n th ball has been moved. (+5)
- b) An office has 3 computers. On each day, each computer breaks down with probability .2, independently of each other. Note that more than one computer can break down on a single day. Once a computer breaks down, it is put aside in the service queue. Early next morning, if there are one or more broken computers in the service queue, the office computer technician fixes it but he can fix only one computer a day. We also assume that a fixed computer doesn't break on the same day. Let $X(n)$ be the backlog of computers waiting for service "at the end of" day n . (Hint: on each day, the number of computers breaking down follows a binomial distribution) (+10)
2. (+30) Sam has a mood swing problem. On any given day, he is depressed, normal or energetic. How he feels on one day depends only on how he felt the previous day. On day n , let $X(n)$ be 0 if he is depressed, 1 if he is normal and 2 if he is energetic. (i.e., you can view $X(n)$ as his "energy level.") Suppose $\{X(n) : n \geq 0\}$ is a time-homogeneous Markov chain with the following transition matrix.

$$P = \begin{matrix} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 0 & 3/4 \\ 1 & 0 & 0 \end{bmatrix} \end{matrix}$$

- a) If Sam is depressed today, what is the probability that he will be normal tomorrow and energetic on the following day? (+5)
- b) If Sam is depressed today with probability .4, normal with probability .3 and energetic with probability .3, what is the probability that he will be normal today, energetic tomorrow, and depressed the next day? (+5)
- c) If Sam is depressed today with probability .4, normal with probability .3 and energetic with probability .3, what is $\underline{\pi}(1)$, the probability distribution of $X(1)$, his state tomorrow? (+5)
- d) Find the limiting distribution $\underline{\pi}$ by explicitly solving the appropriate system of equations. Do the math using fractions. (Don't use calculator to solve the system of equations) (+10)
- e) What is the long-run expected fraction of time that he is normal? (+2)
- f) If Sam is depressed today, what is the expected number of days until he becomes depressed again? (+3)

3. (+45) Ernie has only \$10 but needs \$10 more to get a new rubber ducky for \$20. Ernie plans to get the \$10 from his buddy Bert, who has a lot more money, by engaging him in a betting game. On each play, they bet \$5 each; Ernie has probability .5 of losing his \$5 bet to Bert and probability .5 of winning Bert's \$5 bet. They will stop playing as soon as Ernie either doubles his \$10 or loses everything. Let $X(n)$ be the total amount of money Ernie has after the n th bet; let $X(0) = \$10$. This Markov chain has states 0, 5, 10, 15, 20 (dollars).

- a) Give the transition matrix for this Markov chain. (+5)
- b) The 5th power of the transition matrix is given below. What's the probability that after 5 bets, Ernie is still in the game (i.e., he's neither broke nor has doubled his \$10) and has \$5 in his hands? (+5)

$$P^5 =$$

$$\begin{bmatrix} 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.688 & 0.000 & 0.125 & 0.000 & 0.188 \\ 0.375 & 0.125 & 0.000 & 0.125 & 0.375 \\ 0.188 & 0.000 & 0.125 & 0.000 & 0.688 \\ 0.000 & 0.000 & 0.000 & 0.000 & 1.000 \end{bmatrix}$$

- c) Use the given P^5 again to find the probability that after 5 bets, Ernie is still in the game (i.e., he's neither broke nor has doubled his \$10). (+5)
- d) Rewrite the 1-step transition matrix P you gave in (a) (not P^5 above) in canonical form, i.e., in $\begin{bmatrix} I & 0 \\ R & Q \end{bmatrix}$ format. Then circle the submatrices R and Q and identify them with letters. (+5)

The "fundamental matrix" is $(I - Q)^{-1} = \begin{bmatrix} 1.5 & 1 & .5 \\ 1 & 2 & 1 \\ .5 & 1 & 1.5 \end{bmatrix}$. Use this to answer parts below.

- e) Find the mean time to absorption given that Ernie begins with \$10. (+5)
- f) Find F , the 3x2 matrix of absorption probabilities. (+5)
- g) What is the probability that Ernie will be able to buy the new rubber ducky eventually? (+3)
- h) Let T_i be the time to absorption given that the chain starts in a transient state i . By considering different paths, find $P(T_5=1)$, $P(T_5=2)$, and $P(T_5=3)$. [Note that for this part only, chain begins at \$5, not \$10.] (+9; 3 points for each probability)
- i) What is the mean number of visits that the chain makes to state \$5 before absorption, assuming that the chain starts in state \$10? (+3)