6 The one-way layout

6.1.1 Example (Mucociliary efficiency)

### Mucociliary efficiency from the rate of removal of dust in normal subjects, subjects with obstructive airway disease, and subjects with asbestosis.

\[
x \leftarrow c(2.9, 3.0, 2.5, 2.6, 3.2) \# \text{normal subjects}
\]

\[
y \leftarrow c(3.8, 2.7, 4.0, 2.4) \# \text{with obstructive airway disease}
\]

\[
z \leftarrow c(2.8, 3.4, 3.7, 2.2, 2.0) \# \text{with asbestosis}
\]

Do all three groups have the same median?

What's \( \alpha = 0.502 \) level test?

Rejcet \( H_0 \) if \( H \geq 5.643 \)

\( H = 7.71 \)

What's P-value? \( 0.1009 \)

Approximate P-value? \( 0.64 \)

6.1.2 Data

the data consist of \( N = \sum_{j=1}^{k} n_j \) observations, with \( n_j \) observations from the \( j \)th treatment, \( j=1,\ldots,k \).

6.1.3 Assumptions

A1. The \( N \) random variables \( \{X_{1j}, X_{2j}, \ldots, X_{nj}\}, j=1,\ldots,k \) are mutually independent.

A2. For each fixed \( j \), the \( n_j \) random variables are a random sample from a continuous distribution with distribution function \( F_j \).

A3. The distribution function \( F_1, \ldots, F_k \) are connected through the relationship \( F_j(t) = F(t-\tau_j) \), where \( F \) is a continuous distribution function with unknown median and \( \tau_j \) is the unknown treatment effect for the \( j \)th population.

6.1.4 Hypothesis

The null hypothesis is that of no differences among the treatment effects, i.e.

\( H_0: [\tau_1 = \ldots = \tau_k] \).

6.2 Distribution-free test for general alternatives (Kruskal-Wallis)

We consider the general alternative:

\( H_1: [\tau_1, \ldots, \tau_k \text{ not all equal}] \)
Procedure.
1. combine all \( N \) observations and order them from least to greatest
2. Let \( r_{ij} \) denote the rank of \( X_{ij} \) in this joint ranking
3. Set 
\[
R_j = \sum_{i=1}^{n_j} r_{ij} \quad \text{and} \quad R_j = \frac{R_j}{n_j}, j = 1, \ldots, k
\]

The Kruskal-Wallis statistic is given by
\[
H = \frac{12}{N(N+1)} \sum_{j=1}^{k} n_j \left( R_j - \frac{N+1}{2} \right)^2
\]
where \( N(N+1)/2 \) is the average rank assigned in the joint ranking. (why?)

To test \( H_0 \) vs \( H_1 \) at the \( \alpha \) level of significance,

Reject \( H_0 \) if \( H \geq h_{\alpha} \) where the constant is chosen to make \( P(\text{type I error}) = \alpha \). See Table A.12.

6.2.1 Large sample approximation
When \( H_0 \) is true, the statistic \( H \) has, as \( \min(n_1, \ldots, n_k) \) tends to infinity, an asymptotic \( \chi^2 \) distribution with \( k-1 \) degrees of freedom. The procedure is approximated by

Reject \( H_0 \) if \( H \geq \chi^2_{k-1,\alpha} \). Use Chart A.2.

* Sample size not in order? Re-order them!

6.2.2 R and Minitab Computation
It is implemented in both R and Minitab
[kruskal.test]
Using \( \chi^2 \) table or chart can be tricky.
SAS: WILCOXON in the NPAR1WAY procedure

6.3 A distribution-free test for ordered alternatives (Jonckheere-Terpstra)
A nonparametric test for ordered differences among classes.
It tests the null hypothesis that the distribution of the response variable does not differ among classes.
It is designed to detect alternatives of ordered class differences, which can be expressed as
\[ \tau_1 \leq \tau_2 \leq \cdots \leq \tau_R \quad \text{(or) } \tau_1 \geq \tau_2 \geq \cdots \geq \tau_R, \]

with at least one of the inequalities being strict, where \( \tau_i \) denotes the effect of class \( i \).

For such ordered alternatives, the Jonckheere-Terpstra test can be preferable to tests of more general class difference alternatives, such as the Kruskal-Wallis test.

We calculate \( k(k-1)/2 \) Mann-Whitney counts

\[ U_{uv} = \sum_{i=1}^{n_u} \sum_{j=1}^{n_v} \phi(X_{iu}, X_{jv}), 1 \leq u < v \leq k \]

and compute Jonckheere-Terpstra statistic as

\[ J = \sum_{u=1}^{n-1} \sum_{v=2}^{k} U_{uv} \]

Reject \( H_0 \) if \( J \geq j_\alpha \)

Use Table A.13

Large-Sample Approximation: As \( \min(n_1, \ldots, n_k) \) tends to infinity, the standardized version goes to \( N(0,1) \)

### 6.3.1 Computation

Not implemented in R nor in Minitab. In R, you can run something like:

```r
dat <- number
grp <- as.ordered(information)

jkstat <- function(x, g) {
  foo <- (sign(outer(x, x, "-")) + 1) / 2
  sum(foo[g[row(foo)] > g[col(foo)])
}

print(jstat <- jkstat(dat, grp))

nsim <- 1e4 - 1
jsim <- double(nsim)
for (i in 1:nsim) {
  datsim <- sample(dat, length(dat))
  jsim[i] <- jkstat(datsim, grp)
}

phat <- mean(jsim >= jstat)
(nsimg * phat + 1) / (nsim + 1)
nsim / (nsim + 1) * sqrt(phat * (1 - phat) / nsim)
proc.time()
```
6.4 A distribution-free tests for umbrella alternatives (Mack-Wolfe)

The class of umbrella alternatives:

\[ H_3: \tau_1 \leq \tau_2 \leq \ldots \leq \tau_{p-1} \leq \tau_p \geq \tau_{p+1} \geq \ldots \geq \tau_k \] with at least one strict inequality

6.5 And more! (though not quite mainstream)

http://mason.gmu.edu/~csutton/handwch6657.html

Google Mack-Wolfe and Jonckheere-Terpstra and look at some sites.

6.5.1 Computer implementation

Very rarely implemented in standard statistical computer packages
7 The two-way layout

## Comparison of three methods ("round out", "narrow angle", and
## "wide angle") for rounding first base. For each of 18 players
## and the three method, the average time of two runs from a point on
## the first base line 35ft from home plate to a point 15ft short of
## second base is recorded.

```
> RoundingTimes
Round Out Narrow Angle Wide Angle
1    5.40     5.50     5.55
2    5.85     5.70     5.75
3    5.20     5.60     5.50
4    5.55     5.50     5.40
5    5.90     5.85     5.70
6    5.45     5.55     5.60
7    5.40     5.40     5.35
8    5.45     5.50     5.35
9    5.25     5.15     5.00
10   5.85     5.80     5.70
11   5.25     5.20     5.10
12   5.65     5.55     5.45
13   5.60     5.35     5.45
14   5.05     5.00     4.95
15   5.50     5.50     5.40
16   5.45     5.55     5.50
17   5.55     5.55     5.35
18   5.45     5.50     5.55
19   5.50     5.45     5.25
20   5.65     5.60     5.40
21   5.70     5.65     5.55
22   6.30     6.30     6.25

> friedman.test(RoundingTimes)
boxplot(data.frame(RoundingTimes))
boxplot(data.frame(t(RoundingTimes)))
kruskal.test(data.frame(RoundingTimes))
```

Friedman rank sum test

data:  RoundingTimes
Friedman chi-squared = 11.1429, df = 2, p-value = 0.003805

An experimental design involving two factors, each with at two or more levels
Treatment factor vs blocking factor
A randomized block design
All treatments: treatment(s) vs control (baseline)
# of bservations in each 'cell' (treatment-block combination) with 0, 1 (complete), >1 (replications)
Data:
The data consist of \( N = \sum_{i=1}^{n} \sum_{j=1}^{k} c_{ij} \) observations, with \( c_{ij} \) observations from the combination of \( i^{th} \) block and \( j^{th} \) treatment \([ (i,j)^{th} \text{ cell} ] \) for \( i=1, \ldots, n \) and \( j=1, \ldots, k \).

\( X_{ijt}, a=1, \ldots, c_{ij} \)

Assumptions:
A1. The \( N \) random variables are mutually independent
A2. For each fixed \( (i,j) \), the \( c_{ij} \) random variables \( X_{ijt} \) are a random sample from a continuous distribution \( F_{ij} \)
A3. The distribution functions are connected through:
\[ F_{ij}(u) = F(u - \beta_i - \tau_j), \text{ i.e.} \]

\[ X_{ijt} = \theta + \beta_i + \tau_j + e_{ijt} \]
overall mean + block effect + treatment effect + noise (from a distribution with median 0)

Hypothesis:
H0: \( \tau_1 = \ldots = \tau_k \)

Underlying distributions within each block are the same.
A randomized complete block design: one obs per treatment-block combination

**What's the treatment and block in the example dataset?**

7.1 *A distribution-free test for general alternatives in a randomized complete block design (Friedman)*

H1: \( \tau_1, \ldots, \tau_k \) are not all equal

7.1.1 Procedure

Order the \( k \) observations within each block and let \( r_{ij} \) be the rank of \( X_{ij} \) in the joint ranking in the \( i^{th} \) block. Set \( R_j \) and \( R_{.j} \) as before.

\[ R_j = \sum_{i=1}^{n} r_{ij} \text{ and } R_{.j} = \frac{R_j}{n}, j=1,\ldots,k \]

Then Friedman statistic is

\[ S = \frac{12n}{k(k+1)} \sum_{j=1}^{k} \left( R_j - \frac{k+1}{2} \right)^2 \]

The test of significance level \( \alpha \) rejects H0 if \( S \geq s_{\alpha} \). (Use Table A.22)

* Compute \( S \) for the following subset of the data:
> RoundingTimes

<table>
<thead>
<tr>
<th></th>
<th>Round Out</th>
<th>Narrow Angle</th>
<th>Wide Angle</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.40</td>
<td>5.50</td>
<td>5.55</td>
</tr>
<tr>
<td>2</td>
<td>5.85</td>
<td>5.70</td>
<td>5.75</td>
</tr>
<tr>
<td>3</td>
<td>5.20</td>
<td>5.60</td>
<td>5.50</td>
</tr>
<tr>
<td>4</td>
<td>5.55</td>
<td>5.50</td>
<td>5.40</td>
</tr>
<tr>
<td>5</td>
<td>5.90</td>
<td>5.85</td>
<td>5.70</td>
</tr>
</tbody>
</table>

* Don’t confuse with Kruskal-Wallis statistic

\[ H = \frac{12}{N(N+1)} \sum_{j=1}^{k} n_j \left( R_j - \frac{N+1}{2} \right)^2 \]

* What’s the value of Kruskal statistic and P-value (if we don’t consider this block effect)?

* Experimental design: make an effort to *randomize* (how?)

### 7.1.2 Large-sample approximation

When \( H_0 \) is true, as \( n \) tends to infinity, \( S \sim \chi^2(k-1) \) approximately.

Using the approximation, we reject \( H_0 \) if \( S \geq \chi^2_{k-1, \alpha} \) (Chart A.2)

* Why consider blocks?
  - To compare apples and apples.

* The method works under a slightly weaker condition

* Connection to normal theory test: if we apply the usual two-way layout F test to the ranks instead of the actual observations, we get something proportional to \( S \).

* In the special case of \( k=2 \) treatments, the procedures are identical to the two-sided sign test of section 3.4 (why?)

### 7.2 And more

* Extensions to the cases when we have replications, missing values, etc.
* Extensions when we considered ordered alternatives etc.