2 Probability

2.1 Introduction

- Layman's definition of "Probability": a measure of one's belief in the occurrence of a future event
- "Random" or "stochastic" events : e.g. the blood pressure of a person at a given point in time; the lifetime of a light bulb – cannot be predicted with certainty but the relative frequency in a long series of trials is often stable
- Relative frequency concept of probability: The fraction of heads in a long series of fair coin tossing trials would be near .5 with a fair measure of confidence
  - A gambler bets on the occurrence of a head on a single coin toss if he believes the probability of the head is larger, say 90%.
  - All people are gamblers in many respects, researchers and investors alike betting on the outcome of a single flip of a symbolic coin (research project; success of a company)
- The relative frequency concept of probability is not enough- we need a rigorous definition of probability.
- On a passing note, there are other notions of probability, e.g. "subjective probability" (Bayesian statistics). We're called "frequentist" in comparison. ☺

2.2 Probability and inference

- Consider a gambler who wishes to make an inference concerning the balance of a die
  - The conceptual population of interest is the set of numbers that would be generated if the die were rolled infinitely many times. If the die were balanced, $1/6^{th}$ of the measurements in this population would be 1s, 2s, and so on. (Histogram?)
  - The gambler's hypothesis is that "the die is balanced" Do the observations from nature contradict the theory?
  - A sample of ten tosses is selected from the population by rolling the die ten times. All ten tosses result in 1s.
  - It is not 'impossible' to get the result with a balanced die but it is highly 'improbable'.
  - OK, the probability of the observed sample is very small. But how small? I.e. How strong is the evidence against the 'null hypothesis' if we have four 1s out of ten tosses?
  - Evaluating the probability of such events accurately is critical for such gray area.
- Intuitive evaluations of probabilities often lead to grossly incorrect answers.

- For example, if there are 20 people in a room, how likely it is that there would be two or more persons with the same birthday?
- We need a theory of probability that will permit us to calculate the probability of observing specified outcomes assuming that our hypothesized model is correct.

2.3 A review of set notation

- Capital letters A, B, C,… denote sets of points
- We write A= $\{a_1,a_2,a_3\}$ if the elements in the set A are a_1, a_2, and a_3.
- Basic definitions and notations
  - Let S be the 'universal set', the set of all elements under consideration
  - 'Subset': $A\subset B$
  - 'The null/empty set': $\varnothing$ - a subset of every set
  - 'Venn Diagrams' portray relationships b/w sets
  - The 'union' : $A \cup B$ ("or")
  - The 'intersection': $A \cap B$ ("and")
  - The 'complement' of A, a subset of S: $\bar{A}$ ("not"). $\bar{A} \cup A=S$. We also will write A'.
  - Two sets are 'disjoint' or 'mutually exclusive' if $A\cap B=\varnothing$. A and A' are always disjoint
- Example: Let S be the set of all possible numerical observations for a single toss of a die, i.e. S= {1,2,3,4,5,6}. For A={1,2}, B={1,3}, C={2,4,6}, What are $A\cup B$, $A\cap B$, and $\bar{A}$? Are B and C disjoint? How about A and C?
- Distributive laws: (Venn diagram)
  - $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$
  - $A\cup (B\cap C) = (A\cup B) \cap (A\cup C)$
- DeMorgan's laws: (Venn diagram)
  - $(A\cup B)' = A'\cap B'$
  - $(A\cap B)' = A'\cup B'$
- HW: In chapter 2, do exercises 1, 2, 3
- Keywords: set notations; set algebra; Venn diagram; distributive/DeMorgan's laws

2.4 A probabilistic model for an experiment: the discrete case

- Definition 2.1. An "experiment" is the process by which an observation is made.
  - E.g. Coin tossing
  - E.g. Bacteria counting -determining the number of bacteria per cm^3 in a portion of processed food.

  - Other examples?
- When an experiment is performed, it results in one or more outcomes, called "events"
  - In bacteria counting example,
    - A: exactly 110 bacteria are present
    - B: more than 200 bacteria are present
  - In coin tossing example
    - A: observe an odd number
    - B: Observe a number less than 5
    - E_1: observe a 1.
    - E_2: observe a 2.
    - E_3, …, E_6 defined similarly.
  - Note that if you observe A, you will have observed E_1, E_3 or E_5. The former, a "compound event", can be decomposed into three other events which can't be further decomposed, called "simple events".
  - Connecting things to set theory, we associate a distinct point called a "sample point" with each and every simple event associated with and experiment.
- Definition 2.2. A "simple event" is an event that cannot be decomposed. Each simple event corresponds to one and only one "sample point." We will use the letter E with a subscript to denote a simple event or the corresponding sample point.
  - A simple event = a set consisting of a single sample point associated with the event.
- Definition 2.3. The "sample space," denoted by S, associated with an experiment is the set consisting of all possible sample points.
  - E.g. $S=\{E_1, \ldots, E_6\}$ for coin tossing experiment
  - E.g. $S=\{E_0, E_1, E_2, \ldots\}$ for the bacteria counting experiment
- Definition 2.4. A "discrete sample space" is one that contains either a finite or a countable number of distinct sample points.
  - All distinct simple events correspond to mutually exclusive sets of simple events and are thus mutually exclusive events
  - E.g. in a single toss, you cannot observe a 1 and a 2 at the same time.
- For experiments with discrete sample spaces, compound events can be viewed as unions of the sets of sample points corresponding to the appropriate simple events.
  - E.g. in coin tossing example,
    - $A=\{E_1, E_3, E_5\}$ or $A= E_1\cup E_3\cup E_5$.
    - $B=\{E_1, E_2, E_3, E_4\}$ or $A= E_1\cup E_2\cup E_3\cup E_4$.

- Definition 2.5. An "event" in a discrete sample space S is a collection of sample points (any subset of S)
  - Q. What's the event B in the bacteria counting example?
  - A probability model for an experiment with a discrete sample space can be constructed by assigning a numerical probability to each simple event in the sample space S.
  - We want the numbers in a way consistent with the relative concept of probability
- Axioms for probability: Suppose S is a sample space associated with an experiment. To every event A in S, we assign a number P(A), called the "probability of A" so that the following holds:

  Axiom 1: $P(A) \geq 0$

  Axiom 2: $P(S) = 1$

  Axiom 3: If $A_1, A_2, A_3, \ldots$ form a sequence of pairwise mutually exclusive events in S (that is $A_i \cap A_j = \varnothing$ if $i \neq j$), then

  $P(A_1 \cup A_2 \cup A_3 \cup \ldots) = \Sigma_{i=1,2,\ldots} P(A_i)$
  - Can you explain each axiom in terms of the relative frequency?
  - An assignment of probabilities should satisfy these conditions. But they don't tell us how to assign specific probabilities to events.
- For discrete sample spaces, it suffices to assign probabilities to each simple event.
  - E.g. balanced die-tossing example. $P(E_i)=1/6$ for $i=1,\ldots,6$. Check all axioms are met for this example.
- Example 2.1: a manufacturer has five seemingly identical computer terminals available for shipping. Two of the five are defective. A particular order is filled by randomly selecting two of the five that are available.
  - List the sample space for the experiment
  - Let A denote the event that the order is filled with two nondefective terminals. List the sample points in A.
  - Construct a Venn diagram fro the experiment that illustrates A
  - Assign probabilities to the simple events that are consistent with the 'random' nature of the experiment and obeys the axioms.
  - Find the probability of event A.
- There are experiments for which the sample space is not countable and hence is not discrete.
  - E.g. the experiment where the blood glucose level of a diabetic patient is measured.

  - The sample space for this experiment is an interval of real numbers and any such interval contains an uncountable number of values.
  - Will be discussed in Chapter 4.
- HW: 2.8, 9, 12, 16, 17
- Keywords: experiment; simple event; sample point; (discrete) sample space; event; axioms for probability
- Finding the probability of an event defined on a finite or countable sample space can be approached in two ways:
  - The sample-point method
  - The event-composition method