Practice Problems

3.123 a) $E[Y] = \frac{d}{dt} m(t)\Big|_{t=0} = \frac{1}{6} e^{t} + \frac{4}{6} e^{2t} + \frac{9}{6} e^{3t}\Big|_{t=0} = \frac{7}{3}$

b) $E[Y^2] = \frac{d^2}{dt^2} m(t)\Big|_{t=0} = \frac{1}{6} + \frac{8}{6} + \frac{27}{6} = 6$

$V(Y) = 6 - (\frac{7}{3})^2 = \frac{5}{9}$

c) Since $m(t) = E[e^{tY}]$; Y must take only the values $Y = 1, 2, 3$ with probabilities $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}$ respectively.

3.155 Let Y_1 = # def. out of 5 from line I
Let Y_2 = # def. out of 5 from line II

Then $Y_1 \sim Bin(5, p)$ and $Y_2 \sim Bin(5, p)$, where p is the common probability of def.
Also, $Y_1 + Y_2 \sim Bin(10, p)$.

$$P(Y_1 = 2 \mid Y_1 + Y_2 = 4) = \frac{P(Y_1 = 2 \cap Y_1 + Y_2 = 4)}{P(Y_1 + Y_2 = 4)}$$

$$= \frac{P(Y_1 = 2)\, P(Y_2 = 2)}{P(Y_1 + Y_2 = 4)} = \frac{\binom{5}{2} p^2 q^3 \binom{5}{2} p^2 q^3}{\binom{10}{4} p^4 q^6}$$

$$= \frac{\binom{5}{2}\binom{5}{2}}{\binom{10}{4}} = .476$$

Note that the end probability does not depend on the value of p.

3.159 a) Let X be the number of bacteria colonies in one-cubic-centimeter sample. Then X has a Poisson distribution with $\lambda = 2$. Now let Y be the number of the four one-cubic-centimeter samples that have one or more bacterial colonies. Note that $Y \sim Bin(n=4, p)$, and

$p = P($a sample contains one or more colonies$)$
$= P(X \geq 1) = 1 - P(X=0) = 1 - .135 = .865$

Thus
$P($at least one sample contains one or more colonies$)$
$= P(Y \geq 1) = 1 - P(Y=0) = 1 - \binom{4}{0}(.865)^0(.135)^4 = .9997$

b) We need to find n, such that
$P(Y \geq 1) = 1 - P(Y=0) = 1 - \binom{n}{0}(.865)^0(.135)^n = .95$
or $(.135)^n = .05$
which implies $\log(.135)^n = n \log(.135) = \log(.05)$
so $n = \log(.05)/\log(.135) = 1.496 \doteq 2$ conservative

3.173 a) $p(10) = \dfrac{\binom{40}{10}\binom{60}{10}}{\binom{100}{20}} = .119$

b) binomial approximation with
$p = \frac{r}{N} = \frac{40}{100} = .4$ and $n = 20$
$p(10) = \binom{20}{10}(.4)^{10}(.6)^{10} = P(Y \leq 10) - P(Y \leq 9)$
$= .872 - .755 = .117$

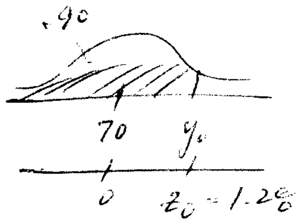
4.128 (optional) a) $F(y) = \int_{-1}^{y} \frac{2}{\pi(1+t^2)}\,dt = \frac{2}{\pi}\left[\tan^{-1}(t)\right]_{-1}^{y}$

$= \frac{2}{\pi}\left[\tan^{-1}(y) - \tan^{-1}(-1)\right] = \frac{2}{\pi}\left[\tan^{-1}(y) - \left(-\frac{\pi}{4}\right)\right]$

$$= \begin{cases} \frac{2}{\pi}\tan^{-1}(y) + \frac{1}{2} & -1 \le y \le 1 \\ 1 & y > 1 \\ 0 & y < -1 \end{cases}$$

b. $E[Y] = \int_{-1}^{1} \frac{2y}{\pi(1+y^2)}\,dy = \frac{1}{\pi}\left[\log(1+y^2)\right]_{-1}^{1} = \frac{1}{\pi}(\log 2 - \log 2) = 0$

4.129 $Y \sim N(70, 12^2)$ find y_0 such that $P(Y < y_0) = .90$



$y_0 = \mu + \sigma z_0 = 70 + 12(1.28)$

$y_0 = 85.36$

4.133 a) The variable factor of $f(y)$ is that of a gamma density with $\alpha = 2$ and $\beta = \frac{1}{2}$. Hence

$c = \frac{1}{\Gamma(\alpha)\beta^\alpha} = \frac{1}{\Gamma(2)\left(\frac{1}{2}\right)^2} = \frac{4}{1!} = 4$

b.) Since $Y \sim \text{Gamma}(2, \frac{1}{2})$

$E[Y] = \alpha\beta = 1 \quad V(Y) = \alpha\beta^2 = 2\left(\frac{1}{4}\right) = \frac{1}{2}$.

c) $m(t) = \frac{1}{(1-\beta t)^\alpha} = \frac{1}{\left[1 - \left(\frac{t}{2}\right)\right]^2} = \left(1 - \frac{t}{2}\right)^{-2}$

4.142 Y = time it takes to interview a job appliant

$f(y) = 2e^{-2y} \quad y > 0$

The second applicant we have to wait only if the time to interview the first applicant exceeds 15 minutes. Hence

$P(\text{wait}) = P\left(Y > \frac{1}{4}\right) = \int_{1/4}^{\infty} 2e^{-2y}\,dy = -e^{-2y}\Big]_{1/4}^{\infty}$

$= e^{-1/2} = .61$

4.152 For $m = 2$,

$$E(Y) = \int_0^\infty \frac{2y^2 e^{-y^2/\alpha}}{\alpha}\, dy$$

Let $z = y^2$. Then $dz = 2y\, dy$ and

$$E(Y) = \int_0^\infty \frac{\sqrt{z}\, e^{-z/\alpha}}{\alpha}\, dz = \int_0^\infty \frac{z^{1/2} e^{-z/\alpha}}{\alpha}\, dz$$

which, when the proper constant is added, will be the integral of the density function of a gamma random variable with parameters $\frac{3}{2}$ and α. Then

$$E(Y) = \frac{\alpha^{3/2}\Gamma\left(\frac{3}{2}\right)}{\alpha} \int_0^\infty \frac{z^{1/2} e^{-z/\alpha}}{\Gamma\left(\frac{3}{2}\right)\alpha^{3/2}}\, dz = \frac{\alpha^{3/2}\Gamma\left(\frac{3}{2}\right)}{\alpha} = \alpha^{1/2}\Gamma\left(\tfrac{3}{2}\right) = \alpha^{1/2}\left(\tfrac{1}{2}\right)\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{(\alpha\pi)^{1/2}}{2}$$

where $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ is shown in exercise 4.162.

Again using the transformation $z = y^2$, we find

$$E(Y^2) = \int_0^\infty \frac{2y^3 e^{-y^2/\alpha}}{\alpha}\, dy = \int_0^\infty \frac{z e^{-z/\alpha}}{\alpha}\, dz = \frac{\Gamma(2)\alpha^2}{\alpha} \int_0^\infty \frac{z e^{-z/\alpha}}{\Gamma(2)\alpha^2}\, dz = \alpha$$

so that $V(Y) = \alpha - \left[\alpha^{1/2}\Gamma\left(\frac{3}{2}\right)\right]^2 = \alpha\left\{1 - \left[\Gamma\left(\frac{3}{2}\right)\right]^2\right\} = \alpha\left[1 - \frac{\pi}{4}\right]$

See Exercise 4.162.

4.153 The density for Y, the life length of a resistor in thousands of hours, is

$$f(y) = \frac{2y e^{-y^2/10}}{10}, \qquad 0 \le y < \infty$$

a. $P(Y > 5) = 1 - P(Y \le 5) = 1 - \int_0^5 \frac{2ye^{-y^2/10}}{10}\, dy = 1 - \left[-e^{-y^2/10}\right]_0^5$

$= 1 - [-e^{-2.5} + 1] = e^{-2.5}$

b. Let X be the number of resistors that burn out prior to 5000 hours. Then X is a binomial random variable with $n = 3$ and $p = 1 - e^{-2.5}$. Thus,

$$P(X = 1) = \binom{3}{1}(1 - e^{-2.5})(e^{-2.5})^2 = .0186$$

4.154 a When $m = 1$

$$f(y) = \frac{1}{\alpha} e^{-y/\alpha} \text{ for } 0 \le y < \infty,\ \alpha > 0$$

which is the density for an Exponential(α) random variable.

b.

$$\begin{aligned} E(Y) &= \int_0^\infty \frac{m}{\alpha} y^m e^{-y^m/\alpha}\, dy \\ &= \int_0^\infty \frac{1}{\alpha} u^{1/m} e^{-u/\alpha}\, du \ \text{ (after making the substitution } u = y^m) \\ &= \alpha^{1/m}\Gamma(1 + 1/m) \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_0^\infty \frac{m}{\alpha} y^{m+1} e^{-y^m/\alpha}\, dy \\ &= \int_0^\infty \frac{1}{\alpha} u^{2/m} e^{-u/\alpha}\, du \ \text{ (after making the substitution } u = y^m) \\ &= \alpha^{2/m}\Gamma(1 + 2/m) \end{aligned}$$

Therefore,

$$V(Y) = \alpha^{2/m}\{\Gamma(1 + 2/m) - \Gamma^2(1 + 1/m)\}.$$

4.148 (optional) a) $P(Y \le 4) = P(X \le \log 4) = P\left(Z \le \frac{\log 4 - 4}{1}\right)$

$= P(Z \le -2.61) = .0045$

b) $P(Y > 8) = P(X > \log 8) = P\left(Z > \frac{\log 8 - 4}{1}\right)$

$= P(Z > -1.92) = 1 - .0274 = .9726$

4.149 (optional) a) $E[Y] = e^{\mu + \sigma^2/2} = e^{3 + 16/2} = e^{11}(10^{-2} g) = 598.74 g$

$V(Y) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) = e^{6+16}(e^{16} - 1) = e^{22}(e^{16} - 1)$

$= (e^{38} - e^{22})(10^{-4} g^2) = 3.1856 \times 10^{12}$

b) with $k = 2$, consider the interval

$E[Y] \pm 2\sqrt{V(Y)}$

$(598.74 \pm 3{,}569{,}637.4) = (-3{,}569{,}038.7,\ 3{,}570{,}236.1)$

Since weights are positive, give as the answer

$(0,\ 3{,}570{,}236.1)$

c) $P(Y < 598.74) = P(\log(Y) < 6.3948)$

$= P\left(Z < \frac{6.3948 - 3}{4}\right) = P(Z < .8487) = .8023$

4.156 a) Let Y have an exponential distribution with parameter θ. Then $f(t) = \frac{1}{\theta} e^{-t/\theta}$

and $1 - F(t) = \int_t^\infty \frac{1}{\theta} e^{-y/\theta}\, dt = -e^{-y/\theta}\Big|_t^\infty = e^{-t/\theta}$.

so $r(t) = \frac{f(t)}{1 - F(t)} = \frac{1}{\theta}$ which is constant.

b) from 4.152, $1 - F(t) = \int_t^\infty \frac{m y^{m-1} e^{-y^m/\alpha}}{\alpha}\, dy$.

let $z = y^m$ so $dz = m y^{m-1} dy$ then

$1 - F(t) = \int_{t^m}^\infty \frac{e^{-z/\alpha}}{\alpha}\, dz = -e^{-z/\alpha}\Big|_{t^m}^\infty = e^{-t^m/\alpha}$

so $r(t) = \frac{m t^{m-1}}{\alpha}$ which is increasing in t for $m > 1$.