The Gamma Function and Gamma Family of Distributions

Gamma Function

The gamma function $\Gamma(t)$ is defined for $t > 0$ as

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx.$$ 

Integrating by parts, one can show that

$$\Gamma(t) = \left[ e^{-x}x/t \right]_0^\infty + (1/t)\Gamma(t + 1) = 0 + (1/t)\Gamma(t + 1).$$

Hence $\Gamma(t + 1) = t \Gamma(t)$.

Thus, if we know values of the function in $(0, 1]$, we can find corresponding values in $(1, 2]$; then in $(2, 3]$, and so on.

The $\Gamma$-function is undefined at $t = 0$.

Although we will not have use for $\Gamma(t)$ at negative values of $t$, we note that this same relationship can be used to extend the function to negative values, except for negative integer values where it is undefined. See the plot below.
Consider the values of the Γ-function at positive integer values of $t$:

It is easy to see that $\Gamma(1) = 1 = 0!$, and so

$\Gamma(2) = \Gamma(1) = 1 = 1!,$

$\Gamma(3) = 2\Gamma(2) = 2 = 2!,$

$\Gamma(4) = 3\Gamma(3) = 3!,$

and so on.

Thus, for any positive integer $t$,

$\Gamma(t) = (t - 1)!.$

In this sense, the Γ-function can be regarded as

a generalization of factorials to noninteger values.
Setting $t = z^2/2$ in the integral over the positive half line of the standard normal density function $\varphi(z)$, we see that $\Gamma(1/2) = \pi^{1/2} = 1.77245$, from which we can get "half-integer" values of the $\Gamma$-function: $\Gamma(3/2) = 0.88623$, etc.

[Note: For $t > 0$, R gives $t \approx 1.461632$ as the point where the (locally rather flat) function attains its minimum value $\approx 0.8856032$.]
For positive values of the parameters $\alpha$ and $\beta$, the *gamma family of probability distributions* has the density function $f(y) = K \, y^{\alpha - 1} e^{-y/\beta}$, for $y \geq 0$; $f(y) = 0$ elsewhere.

The constant $K$ that causes this function to integrate to 1 over the positive half line is $K = [\beta^{\alpha} \Gamma(\alpha)]^{-1}$.

For $\beta = 1$, this is obvious from looking at the definition of $\Gamma(\alpha)$, and it can easily be seen for other values of $\beta$ by making the change of variable $x = y/\beta$.

The parameter $\alpha$ governs the shape of the gamma density and $\beta$ is a scale parameter.

Some books and computer languages use the "rate" parameter $\lambda = 1/\beta$.

For example, in R parameterization with the rate is the default: `pgamma(1/2, 2, 5)` or `pgamma(1/2, 2, rate=5)` returns $P\{X \leq 1/2\} = 0.7127025$, where $X$ has a gamma distribution with $\alpha = 2$ and $\lambda = 1/\beta = 5$.

We get the same result with `pgamma(1/2, 2, scale=.2)`.
The moment generating function of a gamma distribution is 
\[ m(t) = (1 - \beta t)^{-\alpha}. \]

From the mgf it is easy to see that 
the sum of \( r \) independent exponential random variables, 
each with mean \( \beta \) (or rate \( \lambda = 1/\beta \)), has a gamma density with 
shape parameter \( r = \alpha \) and 
scale parameter \( \beta \).

If \( X \sim \text{GAMMA}(\alpha, \beta) \), then 
\[ E(X) = \alpha \beta \] 
\[ V(X) = \alpha \beta^2. \]

There are two notable subfamilies of the gamma family:

An exponential distribution with mean \( \beta \) is \( \text{GAMMA}(1, \beta) \).

A chi-squared distribution with "degrees of freedom" parameter \( \nu \) is 
\( \text{GAMMA}(\nu/2, 2) \).

As illustrated by the R-code below, the gamma family of distributions 
can take five fundamentally different shapes, depending on \( \alpha \).

- \( 0 < \alpha < 1 \): vertical asymptote near the origin (black)
- \( \alpha = 1 \) (exponential): \( f(0^+) = 1/\beta, \ f'(0^+) = -\beta^{-2} \) (red)
- \( 0 < \alpha < 2 \): \( f(0^+) = 0, \ f'(0^+) = \infty \), mode exists, 1 inflection pt. (green)
- \( \alpha = 2 \): \( f(0^+) = 0, \ f'(0^+) = \beta^{-2} \), mode exists, 1 inflection pt. (blue)
- \( \alpha > 2 \): \( f(0^+) = 0, \ f'(0^+) = 0 \), mode exists, 2 inflection pts. (cyan, violet),
where \( 0^+ \) indicates taking a limit as the argument approaches 0 through positive values.
Exercise: (a) Verify the facts stated in the bulleted points above. (b) Also, when they exist, find the mode (point at which the maximum of the density is achieved) and inflection point(s) — in terms of $\alpha$ and $\beta$. (c) Using $\texttt{ylim=c(0,.5)}$, make a similar graph for $\alpha = 1.5, 2, 3,$ and $4$; check your answers against what you can see in the graph.